The effect of a sudden density change on a slightly non-uniform flow

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The effect of combustion or heating on a slightly non-uniform flow in a tube of constant radius is explored by examining a simplified model, in which a density decrease occurs across a plane actuator-disk normal to the axis of the tube. The flow is considered to be steady, axi-symmetric, incompressible and inviscid and to originate at an injector-plate which is situated upstream of the disk. This paper analyses and discusses the effects caused by the density decrease when the velocity of flow through the injector-plate has a fractional non-uniformity u_e, when there is a curvature θ_e of the injector-plate, and when the density decrease has a fractional non-uniformity δ_2. It is shown that the non-uniformity of the flow far downstream of the disk caused by the disturbances u_e and θ_e is diminished by the density decrease; while that due to the disturbance δ_2 is increased, unless the density decrease occurs close to the injector-plate. The distance separating the injector-plate and actuator-disk is found to be an important parameter. When this distance is small, compared to the radius of the tube, and the density decrease is severe, a small disturbance u_e, θ_e or δ_2 may set up a pressure variation at the injector-plate of many times the inlet dynamic pressure of the flow. It is conjectured that this pressure variation could affect the uniformity of velocity and composition of the flow through the injector-plate, and thus cause the disturbances u_e and δ_2 to grow or diminish.

1. Introduction

In rockets, ramjets and turbo-jet combustion chambers a flow with large density changes occurs. The decrease in density caused by combustion considerably affects the pattern of flow. Such effects have been studied by Scurlock (1948), Tsien (1951), Emmons, Ball & Maier (1952) and others, in connexion with flow through a V-shaped flame-front stabilized at the centre of a duct. In these studies, the effect of the flame-front on the flow is due to the shape of the flame-front rather than to any non-uniformity of the approaching flow.† In practical cases the flow approaching the flame-front is generally non-uniform; for instance in rocket combustion chambers the flow of oxidant and fuel through the injector-plate varies with radial distance.

The effect of combustion or heating on non-uniform flows is explored in this paper by examining a simplified model in which a density change occurs across

† Ball (1951) considered the effect of shear in the flow approaching a V-shaped flame-front.

a plane actuator-disk normal to the main-flow direction. This actuator-disk would represent a flame-front if the non-uniformity of the flow at the front was identical with the flame-speed variation in the approaching flow, caused for instance by non-uniform mixture strength.

In the model examined (see figure 1), a steady, axi-symmetric, incompressible, inviscid and slightly non-uniform flow in a tube of constant radius approaches a density-change actuator-disk normal to the axis of the tube. On the upstream

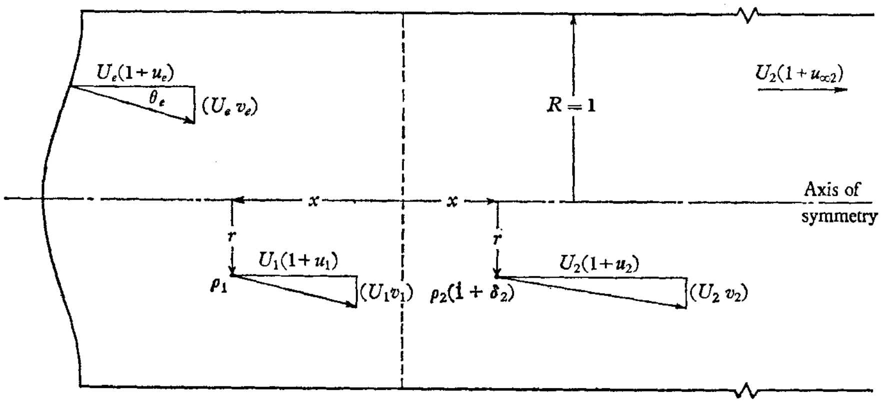

FIGURE 1. Diagram showing flow model and nomenclature.

side of the disk the flow is assumed to be of uniform density. Downstream the density may be either uniform or may vary owing to a slight non-uniformity in the density change at the disk. In accordance with the assumption of inviscid flow there will be no heat transfer, so that, since the flow is also assumed to be incompressible, the density may be considered to be constant along a streamline, except across the disk. Thus the flow on either side of the disk is obtained by a small perturbation of a flow with uniform velocity and density, and the problem may be linearized by neglecting the products of perturbations in the equations defining the flow. The effect of gravity on the flow is neglected. The disk is supposed to be stabilized at a finite distance downstream of a plate through which the fluid is injected, and the flow is assumed to continue for an infinite distance downstream of the disk.

2. Solution for flow on either side of the density-change actuator-disk

The type of flow, downstream of the disk, described above is termed stratified or non-homogeneous; the flow upstream may be regarded as a particular case of such a flow. Long (1953) has shown that the vorticity η in a steady, stratified, plane two-dimensional flow varies along the streamlines as the velocity, of magnitude q, changes, in accord with a relation of the form

$$\eta = H(\Psi)+\frac{1}{\rho}\frac{d\rho}{d\Psi}\frac{q^2}{2}; \tag{1}$$

where ρ is the density, Ψ a stream function such that $d\Psi/dn = -q$ with n measured normal to a streamline, and H is constant along a streamline. Equation (1) is equation (8) of Long (1953) with η replacing $(-\nabla^2\Psi)$, q^2 replacing $(\nabla\Psi)^2$, and the term due to gravity omitted. The equation is also applicable to axi-symmetric flow with the stream function defined by $d\Psi/dn = -qr$, because of the similarity between axi-symmetric and plane two-dimensional flow. The vorticity η is then the circumferential component.

In the problem considered in this paper, the flow is axi-symmetric and involves a small perturbation of a uniform flow, with velocity U and density ρ, in the axial direction. We define the velocity components in the axial, x, and radial, r, directions to be $U(1+u)$ and Uv respectively, and the density to be $\rho(1+\delta)$, where u, v and $\delta \ll 1$; then, by substituting in equation (1) and neglecting products of perturbations, one obtains the equation

$$\eta = H + \frac{d\delta}{d\Psi}\,\tfrac{1}{2}U^2. \tag{2}$$

Now U is constant and H and δ depend only on Ψ, so that the vorticity is constant along the streamlines of the perturbed flow, to first order in the perturbations. These streamlines depart only slightly from the streamlines of the unperturbed flow, which are lines of constant radius. Hence in the perturbed flow the vorticity is a function of radius alone to first order in the perturbations, and may be expressed as

$$\eta = U\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right) = -U\frac{\partial u_\infty}{\partial r}, \tag{3}$$

where Uu_∞ is the value of the axial velocity perturbation at a plane where v and $\partial v/\partial x = 0$.

The flow may be considered to be the sum of the unperturbed flow and a perturbation flow, and, since the products of perturbations are being neglected, the perturbation flow may be resolved into a number of parts. Thus if we write $u = u_\infty + u'$, where u' and v both approach zero at the same plane, then equation (3) shows that Uu' and Uv are the velocity components of the irrotational part of the perturbation flow. The rotational part is seen to have axial velocity Uu_∞ and zero radial velocity. Therefore a perturbation stream function ψ, defined by

$$ru' = -\partial\psi/\partial r \quad \text{and} \quad rv = \partial\psi/\partial x, \tag{4}$$

satisfies Laplace's equation, which in axi-symmetric co-ordinates is

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial x^2} = 0. \tag{5}$$

The solution of this equation which satisfies the boundary conditions, $v = 0$ at $r = 0$ and $r = 1$, the radius of the tube containing the flow being taken as the unit of length, is

$$\psi = \sum_{n=1}^{\infty} (A_n e^{k_n x} + B_n e^{-k_n x})\, rJ_1(k_n r)/k_n, \tag{6}$$

where A_n and B_n are arbitrary constants, and k_n are a series of constants determined from

$$J_1(k_n) = 0, \tag{7}$$

so that $k_1 = 3{\cdot}832$, $k_2 = 7{\cdot}016$, $k_3 = 10{\cdot}174$, etc. Therefore the perturbation flow may be considered as the sum of a series of Fourier–Bessel harmonics. The nth harmonic of the perturbation stream-function, and of the radial velocity, has a radial form defined by the Bessel function $J_1(k_n r)$, which has $(n-1)$ zeros between the tube axis and the tube wall. The factor (in any harmonic) which is independent of the radius r, e.g. $k_n^{-1}\{A_n e^{k_n x} + B_n e^{-k_n x}\}$ in equation (6), will be referred to as the amplitude of the harmonic.

Using equations (4) and (6) the velocity perturbations Uu and Uv in the flow on either side of the density-change actuator-disk are found to be given by expressions of the form

$$u = \sum_{n=1}^{\infty} u_n J_0(k_n r), \tag{8}$$

and

$$v = \sum_{n=1}^{\infty} v_n J_1(k_n r), \tag{9}$$

where the amplitudes u_n and v_n of the nth harmonics are

$$u_n = u_{\infty n} + u'_n \quad \text{with} \quad u'_n = -(A_n e^{k_n x} + B_n e^{-k_n x}) \tag{10}$$

and

$$v_n = (A_n e^{k_n x} - B_n e^{-k_n x}). \tag{11}$$

The amplitude $u_{\infty n}$ of the nth harmonic of u_∞ is defined by equation (8). The quantities u and v will be referred to as the axial and radial velocity perturbation fractions respectively, and u will be used to define the non-uniformity of axial velocity.

An expression for the pressure-perturbation fraction p, that is the static pressure-perturbation divided by the dynamic pressure of the unperturbed flow, may be derived by equating the terms containing perturbations on either side of the equation of motion in the radial direction, and neglecting products of perturbations, from which one obtains

$$-(\tfrac{1}{2}\rho U^2)\frac{\partial p}{\partial r} = (\rho U^2)\frac{\partial v}{\partial x}.$$

Combination of this result with equation (3) gives

$$\frac{\partial u'}{\partial r} = \frac{\partial v}{\partial x} = -\frac{1}{2}\frac{\partial p}{\partial r}, \tag{12}$$

from which it may be concluded that

$$p = -2u', \tag{13}$$

the constant vanishing because, at the plane where $u' = 0$, the flow is parallel to the axis.

3. Application of boundary conditions

The quantities $u_{\infty n}$, A_n and B_n in the expressions for the perturbation fractions are determined by the boundary conditions at the injector-plate, at the density-change actuator-disk, and at an infinite distance downstream. The additional suffixes 1, 2, a, and e will be used to denote conditions upstream of the actuator-disk, downstream of the actuator-disk, at the actuator-disk, and at the injector-plate respectively.

The boundary conditions at the injector-plate are determined by the non-uniformity of the flow through the plate and the geometry of the plate. The plate is assumed to be symmetrical about the axis of the tube, and to be slightly curved. Hence the perturbation of the axial velocity at the plate $U_1 u_e$ arises directly from the non-uniformity of the flow through the plate, and can be defined by

$$u_e = \sum_{n=1}^{\infty} u_{en} J_0(k_n r). \tag{14}$$

The curvature of the plate will produce a radial velocity at the plate, which, as a fraction of U_1, may be written

$$-v_e = \theta_e = \sum_{n=1}^{\infty} \theta_{en} J_1(k_n r), \tag{15}$$

where θ_e is the angle by which the entering flow is deflected inwards from the axial direction due to the curvature of the plate. It is equal to the angle by which the surface of the plate deviates from that of a flat plate. Near the tube wall θ_e must approach zero so that the flow adjacent to the wall is injected parallel to the wall.

The flows upstream and downstream of the density-change actuator-disk are matched at the disk by the conditions of conservation of mass and momentum. From conservation of mass, we derive that

$$\rho_1 U_1(1+u_{a1}) = \rho_2(1+\delta_2)\, U_2(1+u_{a2}).$$

By neglecting the term containing a product of perturbations, and equating the terms corresponding to the unperturbed and the perturbation flows on either side of the equation, one obtains the results

$$\rho_1 U_1 = \rho_2 U_2 \tag{16}$$

and

$$u_{a1} = u_{a2} + \delta_2. \tag{17}$$

Hence, using equation (8), the non-uniformity of the density change may be expanded into harmonics, to give

$$\delta_2 = \sum_{n=1}^{\infty} \delta_{2n} J_0(k_n r). \tag{18}$$

From the conservation, across the disk, of mass and radial momentum, it is apparent that the radial velocity is continuous across the disk, so that, using equation (16),

$$\lambda v_{a1} = v_{a2}, \tag{19}$$

where $\lambda = \rho_2/\rho_1$ defines the ratio of the densities of the unperturbed flow on either side of the actuator-disk.

Conservation of momentum normal to the plane of the disk gives an equation which, on neglecting terms containing products of perturbations, equating the terms corresponding to the perturbation flow on either side of the equation, and using equation (16), becomes

$$\lambda p_{a1} - p_{a2} = 2(2u_{a2} + \delta_2 - 2\lambda u_{a1}),$$

which, with equations (13) and (17), reduces to

$$\lambda u_{\infty 1} - u_{\infty 2} = (1-\lambda)\, u_{a1}. \tag{20}$$

This equation defines the change, at the actuator-disk, of the rotational part of the perturbation flow, u_∞, which is otherwise invariant along the tube, in terms of the conditions at the disk. Thus it leads directly to an expression for the change in the vorticity of the flow at the disk, and the relation of this result to the work of other authors is discussed in § 4.2.

At an infinite distance downstream from the disk, the streamlines are assumed to be parallel. Thus at $x = \infty$, u', v and $\partial v/\partial x = 0$, and $U_2 u_{\infty 2}$ is the only remaining velocity perturbation, so that, from equation (11),

$$A_{2n} = 0. \tag{21}$$

By substitution of the solutions for the velocity perturbation fractions, equations (8) to (11), in the boundary conditions, equations (14), (15), (17), (19) and (20), and by use of equations (18) and (21), expressions for $u_{\infty n}$, A_n and B_n can be obtained in terms of λ, $e^{-k_n a}$, u_{en}, θ_{en} and δ_{2n}, where a is the distance separating the injector-plate and actuator-disk. These expressions are shown in the Appendix, in which $u_{\infty n}$, A_n and B_n are related to u_{en}, θ_{en} and δ_{2n} by influence coefficients. The equations stating the dependence of u, v and p on x, r, $u_{\infty n}$, A_n and B_n are also summarized in the Appendix. Using the Appendix, the flow pattern arising from any combination of the disturbing non-uniformities, u_e, θ_e and δ_2, can be assessed. The number of terms required in the summations for u, v and p will depend on the number of Fourier–Bessel harmonics required to form each disturbing non-uniformity.

4. Discussion of solution

4.1. *Assumption of incompressible flow*

The effect of the compressibility of the fluid will become appreciable when the Mach number of the unperturbed flow downstream of the density-change actuator-disk exceeds a value of about 0·3. The ratio of the Mach numbers of the unperturbed flow on either side of the disk is given by

$$\frac{M_2}{M_1} = \frac{U_2}{U_1}\left(\frac{T_1}{T_2}\right)^{\frac{1}{2}} = \lambda^{-\frac{1}{2}},$$

if the fluid is assumed to be a semi-perfect gas, since then $\rho_1 T_1 = \rho_2 T_2$, for small M_1 and M_2. Hence the solution is adequate for

$$\lambda < 10 M_1^2. \tag{22}$$

If the temperature downstream of the density change were 4500 °K, a very high value for flame temperature, then λ would equal 0·07 and the above condition would require $M_1 < 0{\cdot}08$. Combustion reactions in gas mixtures proceed at Mach numbers much less than this value, so that the assumption of incompressible flow is justified for the flow of gas mixtures normally through an actuator-disk at which combustion takes place.

In rocket combustion chambers, the Mach number of the flow when combustion is completed is determined by the ratio of the cross-sectional areas of the combustion chamber and the nozzle throat, where $M = 1$. If this area ratio equals 3 or more then $M_2 < 0{\cdot}3$ and compressibility effects may be neglected.

4.2. *Vorticity change at density-change actuator-disk*

The change in the vorticity of the flow at the disk is given by equation (20) with equations (3) and (16), from which

$$\eta_2 - \eta_1 = U_1 \frac{(1-\lambda)}{\lambda} \frac{\partial u_{a1}}{\partial r}. \tag{23}$$

Hence the vorticity of the flow is changed at the disk by an amount determined by the radial gradient of axial velocity just upstream of the disk and the density ratio across the disk. The vorticity change is zero if the density change is small, and does not depend directly on any small non-uniformity in the density change at the disk, to first order in the perturbations.

Hayes (1957) has studied the general case of rotational compressible flow through a surface of discontinuity in density. Hayes shows that only the component of vorticity tangential to the surface experiences a change as the flow passes through the surface, and that this change is made up of two parts, both of which depend on the ratio of the densities on either side of the surface. One part of the change is proportional to the gradient along the surface of the mass velocity normal to the surface, and this part reduces to equation (23) for the conditions of the problem considered in this paper. The other part varies with the product of the component of velocity tangential to the surface and its gradient in its own direction, which in this paper is $v(\partial v/\partial r)$ and so is zero to first order. This latter part provided a vorticity change at the flame-fronts studied by the authors cited in § 1. In the papers by these authors, the flame speed, which is the velocity of flow normal to and into a stationary flame-front, and the density upstream of the flame, were considered constant, so that the part of the vorticity change which is important in this paper was identically zero.

4.3. *Case of injector-plate and actuator-disk at an infinite distance apart* ($a = \infty$)

If the injector-plate is at an infinite distance from the actuator-disk, then there is some point between them at which the radial velocity approaches zero. Hence the relevant disturbing non-uniformities in this case are upstream vorticity, represented by $u_{\infty 1}$, and non-uniform density change at the disk, δ_2. The upstream vorticity may be caused by either or both of the disturbances u_e and θ_e at the injector-plate, and the expression for the amplitude of the nth harmonic of $u_{\infty 1}$ is, from the Appendix,

$$u_{\infty 1n} = u_{en} + \theta_{en}. \tag{24}$$

The main results of the analysis in this case do not depend on the *form* of radial variation of the disturbances. For with $a = \infty$, so that $\epsilon_n = e^{-k_n a} = 0$, it is seen from the table in the Appendix that $u_{\infty n}$, A_n and B_n are independent of k_n. Therefore at $x = -\infty$, $x = 0$ and $x = +\infty$, where $e^{k_n x} = 0$, $e^{\pm k_n x} = 1$ and $e^{-k_n x} = 0$ respectively, the amplitude of the velocity perturbation caused by a disturbance consisting of a single harmonic of unit amplitude is the same for all harmonics. Thus the radial *form* of a velocity perturbation is identical at the disk and at an infinite distance upstream and downstream of the disk, even if the disturbance

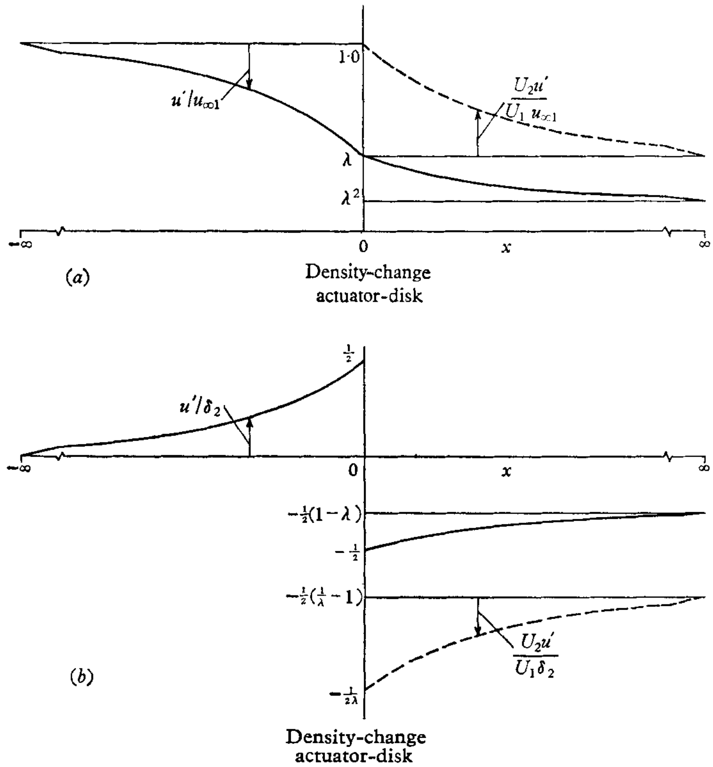

FIGURE 2. Axial velocity perturbations caused when the injector-plate is at an infinite distance upstream of the density-change actuator-disk. The figure is to scale for $\lambda = 0{\cdot}4$. (*a*) ——, $u/u_{\infty 1}$; – – – –, $U_2 u/U_1 u_{\infty 1}$. (*b*) ——, u/δ_2; – – – –, $U_2 u/U_1 \delta_2$.

consists of more than one harmonic. For example the amplitude of the nth harmonic of the axial velocity perturbation fraction just upstream of the actuator-disk is, from equation (10),

$$u_{a1n} = u_{\infty 1n} - A_{1n} - B_{1n},$$

which, using the influence coefficients in the Appendix, becomes, with $\epsilon_n = 0$,

$$u_{a1n} = (u_{en} + \theta_{en}) - (1-\lambda)(u_{en} + \theta_{en}) + \tfrac{1}{2}\delta_{2n},$$

and, with equations (24), (8) and (18), this can be summed to give

$$u_{a1} = \lambda u_{\infty 1} + \tfrac{1}{2}\delta_2. \tag{25}$$

The variation of the axial velocity perturbation fraction, expressed as $u/u_{\infty 1}$, and of the perturbation, expressed as $Uu/U_1 u_{\infty 1}$, in the axial, x, direction along a line of constant radius takes the form shown in figure 2, which shows separately the effects of $u_{\infty 1}$, figure 2 (a), and δ_2, figure 2 (b). The perturbation fraction is indicated by full lines, and the perturbation by full lines upstream of the disk and broken lines downstream. The ordinates of significant points on the curves are shown in terms of the density ratio λ, and the curves are drawn to scale for $\lambda = 0{\cdot}4$. The scale of x is omitted since the various Fourier–Bessel harmonics of a perturbation differ in their manner of variation with x, though the harmonics behave identically at $x = -\infty$, $x = 0$ and $x = +\infty$ as explained above.

At equal distances on either side of the disk the irrotational part of the perturbation Uu' is equal in magnitude and opposite in sign, for either of the disturbances δ_2 and $u_{\infty 1}$. From figure 2 it may be seen that these disturbances cause identical values of Uu' at the disk, and thus everywhere in the flow, for a particular Fourier–Bessel harmonic, if

$$\delta_2 = -2(1-\lambda)\, u_{\infty 1}. \tag{26}$$

Identical values of Uu' lead to identical values of v and p, from equation (12). Hence the radial velocities and pressure perturbations in the flow produced by a disturbance $u_{\infty 1}$ may be identically reproduced by a disturbance δ_2 which satisfies equation (26), and vice versa. The exceptional case $\lambda = 1$ is discussed in § 4.4.4 in connexion with the similar result obtained when a is finite.

From figure 2 (a), for which $u_{\infty 1} \neq 0$ and $\delta_2 = 0$, it is seen that a disturbing non-uniformity of axial velocity $u_{\infty 1}$ is severely attenuated by density decrease, since $u_{\infty 2} = \lambda^2 u_{\infty 1}$; the axial velocity perturbation is also attenuated by the ratio λ. The axial variation of the perturbation is seen to be of very simple form, the perturbation just upstream of the disk being equal to that far downstream, and vice versa.

From figure 2 (b), for which $u_{\infty 1} = 0$ and $\delta_2 \neq 0$, it is seen that a disturbance δ_2, representing a non-uniform density change at the disk, causes an irrotational upstream flow, $u_{\infty 1} = 0$, to become rotational downstream of the disk. The magnitude of the rotational part of the downstream axial velocity perturbation fraction or non-uniformity $u_{\infty 2}$ increases as the density ratio λ decreases, and approaches the maximum value of $(\frac{1}{2}\delta_2)$ as the downstream density approaches zero, $\lambda \to 0$.

4.4. *General case*

4.4.1. *The effect on the results of the radial form of a disturbance*

The distance a separating the injector-plate and the density-change actuator-disk is an important parameter in the problem, and appears in the results of the analysis in the quantity $\epsilon_n = e^{-k_n a}$. Hence the effect of a on the velocity perturbations caused by a disturbance consisting of a particular Fourier–Bessel harmonic depends on the number n of the harmonic. If a disturbance consists of more than one harmonic, then, in general, the radial form of a resulting velocity perturbation will differ at the injector-plate, the disk, and infinity downstream, in contrast to the result obtained in the special case $a = \infty$, § 4.3. The factor $k_n a$, which determines the effect of a on the results for the nth harmonic of a disturbance, is approximately equal to π times the ratio of a and the radial distance between adjacent zeros of the radial velocity perturbation resulting from that harmonic. This rough equality becomes more exact with increase of the value of $(n-1)$, the number of zeros of the radial velocity perturbation between the tube axis and the tube wall.

The effect on the results of a being finite rather than infinite is greatest for the primary disturbance harmonic, $n = 1$, so that in the following discussion the disturbances are considered to consist of this harmonic alone. The results for a disturbance consisting of the nth harmonic alone can be obtained from those for the primary harmonic simply by multiplying the values of a by the ratio (k_1/k_n). The flows resulting from each disturbance acting singly are discussed in the following sections.

4.4.2. *Flow caused by non-uniform axial velocity at the injector-plate (disturbance u_e)*

The variation of (u/u_e) and (Uu/U_1u_e) with x is shown in figure 3 for values of a of 0, $\frac{1}{2}\beta$, β and ∞, where $\beta = k_1^{-1}\ln 2$. The arrangement of this figure is similar to that of figure 2 (a) and the curves from figure 2 (a), for $a = \infty$, are repeated on it. The variation of the axial velocity non-uniformity (u/u_e) is shown by the full lines, and it is seen that as a is reduced so is the attenuation $\{1-(u_{\infty 2}/u_e)\}$ of the disturbance by the density decrease. The attenuation increases as the density ratio λ is decreased, for all values of a except zero. In the limiting case of $a = 0$, the density change occurs immediately after injection and the flow is invariant along the tube. The axial velocity perturbation (Uu/U_1u_e) is shown on figure 3 by full lines upstream of the disk and broken lines downstream. It is seen that the axial velocity perturbation is greater at infinity downstream than at the injector-plate for $a < \beta$, and less for $a > \beta$.

The variation with a of the upstream value of the rotational part of the axial velocity perturbation fraction, $u_{\infty 1}$, is also indicated on figure 3. It is seen that $u_{\infty 1}$ increases as a decreases; therefore, since u is nowhere greater than u_e, $u_1' = (u_1 - u_{\infty 1})$ also increases in magnitude as a decreases. Hence, from equation (13), it is seen that the pressure perturbations upstream of the disk increase in magnitude as a decreases. The radial pressure variation set up at the injector plate by the disturbance is of particular interest, since this variation may alter the uniformity of injection, and thus the disturbance, if the rate of injection of fluid at the plate is pressure controlled, as in liquid rockets. The pressure perturbation fraction at the injector-plate, p_e, is shown on figure 4, plotted against a, for various values of λ. It is seen that for $a < 0{\cdot}1$ and $\lambda < 0{\cdot}2$, conditions which may occur in liquid rockets, then $p_e \gg u_e$. Hence under these conditions a radial pressure variation of many times the inlet dynamic pressure $(\frac{1}{2}\rho_1 U_1^2)$ may be obtained at the injector-plate, even if u_e is small.

The maximum value of the radial velocity occurs at the disk, since u' defines $\partial v/\partial x$, equation (12), and is of constant sign on either side of the disk, changing sign at the disk (see figure 3). The variation of v_{a1}, the radial velocity at the disk as a fraction of U_1, with a and λ is shown in figure 5. Because of the difference in the radial forms of v_{a1} and u_e, the maximum values $\hat{v}_{a1}$ and $\hat{u}_e$ are used for com-

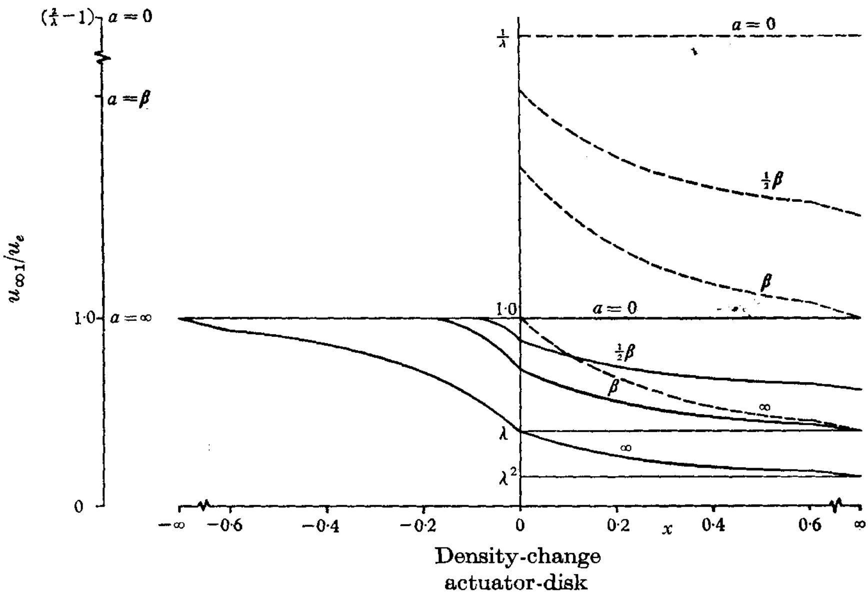

FIGURE 3. Axial velocity perturbations caused by a disturbance u_e of primary form. $\beta = k_1^{-1} \ln 2 = 0{\cdot}181$. The figure is to scale for $\lambda = 0{\cdot}4$. ——, u/u_e; ----, Uu/U_1u_e.

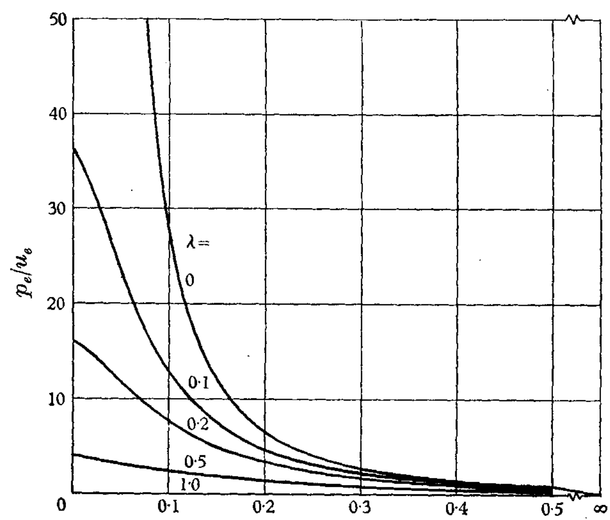

FIGURE 4. Pressure-perturbation fraction at the injector-plate caused by a disturbance u_e of primary form.

parison. In the analysis, it is assumed that $v \ll 1$. From figure 5 it is apparent that this assumption is not valid for $\lambda \ll 0{\cdot}1$ and $0 < a < 0{\cdot}1$, since under these conditions $v_{a1} \gg u_e$. However, this range of conditions is more restricted than those stated above for which $p_e \gg u_e$, so that appreciable radial pressure variations may be caused at the injector-plate, though the radial velocity is everywhere small.

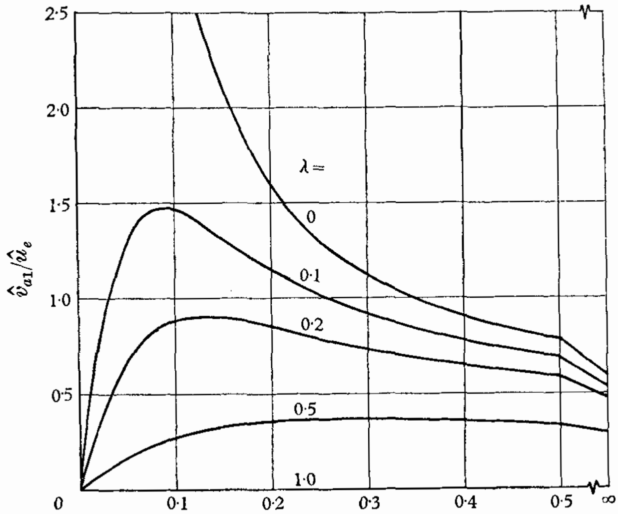

FIGURE 5. Radial velocity perturbation fraction at the density-change actuator-disk caused by a disturbance u_e of primary form. Maximum values are compared.

4.4.3. *Flow caused by a curved injector-plate (disturbance θ_e)*

In the absence of density change a curved injector-plate is seen, from equation (24), to cause a uniform flow to develop a non-uniformity of axial velocity with the same amplitude as θ_e. This non-uniformity will be attenuated in the manner described in § 4.3 by a density decrease occurring at a plane far downstream of the injector-plate. The change in this attenuation when the density decrease takes place at a finite distance downstream of the injector-plate is shown by figure 6, in which the variation with a of $u_{\infty 2}$, the non-uniformity of axial velocity far downstream of the disk, is shown for various values of λ. The attenuation $\{1-(\hat{u}_{\infty 2}/\hat{\theta}_e)\}$ of the disturbance is seen to decrease with a except near $a = 0$, and to increase as the density ratio λ is reduced, for all values of a.

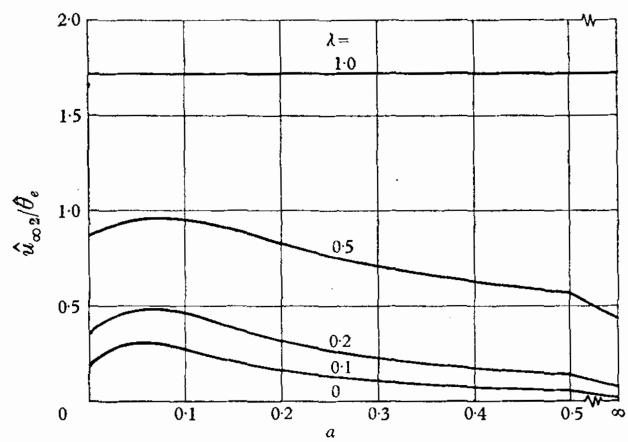

FIGURE 6. Non-uniformity of axial velocity far downstream of the density-change actuator-disk, caused by a disturbance θ_e of primary form. Maximum values are compared.

It may be shown that the velocity perturbation fractions u and v are everywhere of the same order of magnitude as the disturbance θ_e for all values of a and λ. However, an appreciable radial pressure variation may be set up at the injector-plate, as shown by figure 7 where p_e is plotted against a for various values of λ. It is seen that a radial pressure variation of many times the inlet dynamic pressure may be obtained at the injector-plate for $\lambda < 0{\cdot}1$ and $0 < a < 0{\cdot}1$, even if θ_e is small.

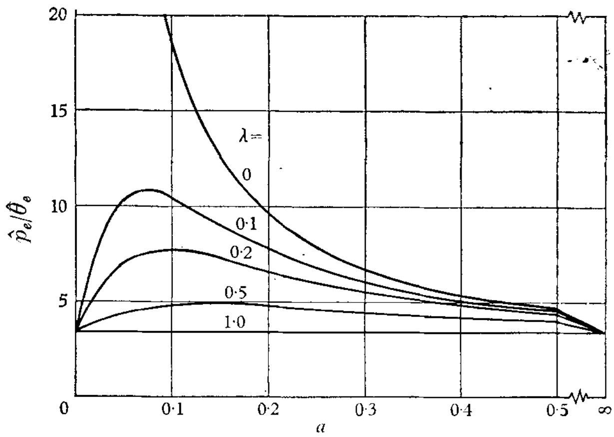

FIGURE 7. Pressure-perturbation fraction at the injector-plate caused by a disturbance θ_e of primary form. Maximum values are compared.

4.4.4. *Flow caused by a non-uniform density change (disturbance δ_2)*

It is apparent from the table of influence coefficients in the Appendix that the values of A_n and B_n, and therefore of the irrotational part of a velocity perturbation, caused by a disturbance δ_2 are identical with those caused by a disturbance u_e, if

$$\delta_2 = -2(1-\lambda)\,u_e. \tag{27}$$

Hence, in view of equation (12), the discussion in §4.4.2 of the radial velocities and pressure perturbations caused by a disturbance u_e applies also to the flow caused by a disturbance δ_2, except when $\lambda = 1$. The exception arises because if there is no density change in the unperturbed flow, the disturbance u_e is invariant along the tube, whereas the disturbance δ_2 necessarily sets up a discontinuity in u at the disk (see equation (17)). However, the radial velocities and pressure perturbations due to δ_2 are small when $\lambda = 1$, so that the conditions for an appreciable pressure variation to be set up at the injector-plate by δ_2 are as stated for u_e. For density change caused by combustion it is possible that such a pressure variation may alter the non-uniformity of the mixture ratio of the injected fluid and thus cause the non-uniformity of the density change to grow or diminish.

The non-uniformity of axial velocity far downstream, $u_{\infty 2}$, caused by this disturbance is shown on figure 8 plotted against a for various values of λ. It is seen that if there is no density change in the unperturbed flow, $\lambda = 1$, then $u_{\infty 2}$ varies from zero when $a = \infty$ to the value of $-\delta_2$ when $a = 0$. The effect of density decrease is to amplify this non-uniformity for $a > \beta$ and attenuate it for $0 < a < \beta$, where $\beta = k_1^{-1} \ln 2 = 0{\cdot}181$. Thus $u_{\infty 2}$ approaches the value of $-\frac{1}{2}\delta_2$ for all values of a as λ approaches zero.

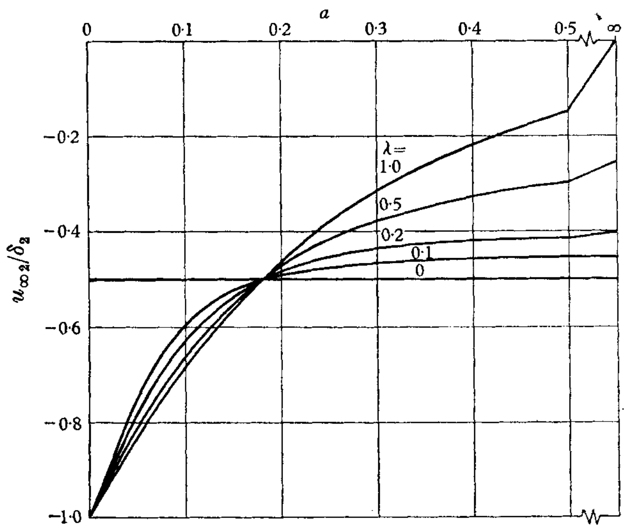

FIGURE 8. Non-uniformity of axial velocity far downstream of the density-change actuator-disk, caused by a disturbance δ_2 of primary form.

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Appendix: summary of solution

Disturbing non-uniformities

$$u_e = \sum_{n=1}^{\infty} u_{en} J_0(k_n r), \quad \theta_e = \sum_{n=1}^{\infty} \theta_{en} J_1(k_n r)$$

and

$$\delta_2 = \sum_{n=1}^{\infty} \delta_{2n} J_0(k_n r).$$

Resulting perturbation fractions

$$u_j = \sum_{n=1}^{\infty} u_{jn} J_0(k_n r), \quad v_j = \sum_{n=1}^{\infty} v_{jn} J_1(k_n r) \quad \text{and} \quad p_j = -2u_j',$$

with

$$u_{jn} = u_{\infty jn} + u'_{jn}, \quad u'_{jn} = -(A_{jn} e^{k_n x} + B_{jn} e^{-k_n x}),$$

$$v_{jn} = A_{jn} e^{k_n x} - B_{jn} e^{-k_n x},$$

where

$$j = 1 \text{ for } -a \leqslant x \leqslant 0, \quad j = 2 \text{ for } 0 \leqslant x \leqslant \infty.$$

Influence coefficients

	u_{en}/E	θ_{en}/E	δ_{2n}/E
$u_{\infty 1n}$	$\dfrac{1}{(1-\lambda)}+\epsilon_n^2$	$\dfrac{1}{(1-\lambda)}-\epsilon_n^2$	$-\dfrac{\epsilon_n}{(1-\lambda)}$
$u_{\infty 2n}$	$\lambda\left\{\dfrac{\lambda}{(1-\lambda)}+\epsilon_n^2\right\}$	$\lambda\left\{\dfrac{\lambda}{(1-\lambda)}+\epsilon_n-\epsilon_n^2\right\}$	$-\dfrac{1}{2}\left\{1-2\dfrac{(1-2\lambda)}{(1-\lambda)}\epsilon_n+\epsilon_n^2\right\}$
A_{1n}	1	$1-\epsilon_n$	$-\dfrac{1}{2(1-\lambda)}$
B_{1n}	ϵ_n^2	$\dfrac{\epsilon_n}{(1-\lambda)}-\epsilon_n^2$	$-\dfrac{\epsilon_n^2}{2(1-\lambda)}$
A_{2n}	0	0	0
B_{2n}	$-\lambda(1-\epsilon_n^2)$	$-\lambda\left\{1-\dfrac{(2-\lambda)}{(1-\lambda)}\epsilon_n+\epsilon_n^2\right\}$	$\dfrac{\lambda(1-\epsilon_n^2)}{2(1-\lambda)}$

Here k_n are the roots of $J_1(k_n) = 0$, thus $k_1 = 3{\cdot}832$, $k_2 = 7{\cdot}016$, $k_3 = 10{\cdot}174$, etc.,

$$\lambda = \rho_2/\rho_1, \quad \epsilon_n = e^{-k_n a} \quad \text{and} \quad E = \{[1/(1-\lambda)]-2\epsilon_n+\epsilon_n^2\}.$$